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WATER WAVES ON A SHALLOW SLOPING BEACH

By

K. O. FRIEDRICHS

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The problem of standing and progressing waves on a beach with a plane bottom has been solved by Miche [1], Lewy and Stoker [2], [3] for slope angles ω which are integral fractions of a right angle,* $\omega = \pi/2n$, $n = 1,2,\ldots$. In the present report, in which various suggestions by J. J. Stoker and H. Lewy are incorporated, a simple asymptotic representation for small angles ω is presented.** It enables us in particular to show that for small slope angles the flow can be described as consisting of waves with a varying wave length, which depends on the depth in the same way as Airy's waves in channels with constant depth. In addition our formulas give the dependence of the amplitude and the phase on the depth for small slope angles. The wave shape calculated on the basis of our formulas for a beach with a 6° slope, see Figures 2 and 3, agrees almost perfectly with that calculated on the basis of the exact theory, see Stoker [3, p. 35].

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*Lewy solves more generally the problem with the slope angle $\omega = p\pi/2n$, $p = 1,3,5,\ldots$, $(p,n) = 1$.

**The derivation of this asymptotic representation from a new form of the solution for slope angles $\omega = \pi/2n$ will be given in a later publication.

1. Mathematically speaking the problem* is that of finding an analytic function $\chi(z)$ of the complex variable $z = x + iy$ defined in the sector $z = re^{i\theta}$, $-\omega \leq \theta \leq 0$, $r \geq 0$, see Figure 1 , which satisfies the boundary conditions

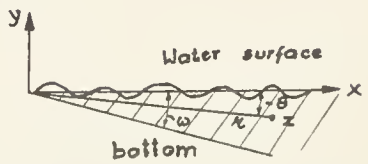

Figure 1
Sloping beach

$$\text{I} \qquad \operatorname{Im}\left(\frac{d}{dz} + i\right)\chi = 0 \quad \text{on the line} \quad \theta = 0,$$

$$\text{II} \qquad \operatorname{Im} e^{-i\omega}\frac{d}{dz}\chi = 0 \quad \text{on the line} \quad \theta = -\omega,$$

remains bounded as r approaches infinity, and becomes at most logarithmically infinite as $r \to 0$.

The function

$$(1) \qquad \chi(z,t) = \chi(z)\cos t$$

is the complex potential function of the flow representing a standing wave. The units of z and t are so chosen that both the circular frequency σ and the acceleration of gravity g have the value one, see Stoker [3, Section 3]. The velocity components u and v in the x and y directions are

*The problem of waves on sloping beaches and the connection of the questions here treated with related questions are explained in an introductory report [4].

derived from the function χ through $u - iv = d\chi/dz$. The boundary condition II expresses the requirement that the water remain attached to the rigid bottom surface $\theta = -\omega$. Condition I results from linearizing the condition that the pressure on the top surface remain constant. We note that the elevation of the free water surface is

$$\eta(x,t) = \mathrm{Re}\, \frac{\partial}{\partial t} \chi(x,t). \tag{2}$$

For details see the two papers [2 and 3] quoted.

2. There are two <u>solutions</u> of the problem. One solution, $\chi^R(z)$, is regular at the origin, $z = 0$, and behaves like $\chi^R_\infty = -iC\, e^{-iz}$ at infinity

$$\chi^R(z) + iC\, e^{-iz} \to 0 \quad \text{as} \quad |z| \to \infty\ , \tag{3}$$

with an appropriate complex constant C, see [3, Sec. 2]. The other solution, $\chi^S(z)$, is singular at $z = 0$, in such a way that

$$\chi^S(z) - 2 \log z \quad \text{is regular at} \quad z = 0, \tag{4}$$

while χ^S behaves like $\chi^S_\infty = C\, e^{-iz}$ at infinity,

$$\chi^S(z) - C\, e^{-iz} \to 0 \quad \text{as} \quad |z| \to \infty\ . \tag{5}$$

The standing waves, given by

$$\chi^R = \chi^R(z)\cos t \quad \text{and} \quad \chi^R = \chi^S(z)\sin t \ ,$$

are evidently 90° out of phase at infinity.

With the aid of these two standing waves one can build up progressing waves traveling toward shore. Such a progressing wave is given by the complex potential

$$(6) \qquad \chi^P(z,t) = \chi^S(z)\cos t + \chi^R(z)\sin t \ ,$$

which, at infinity, behaves like

$$(7) \qquad \chi^P(z,t) \approx C\, e^{-i(z+t)}, \quad z \approx \infty \ ,$$

and thus like a sinusoidal wave traveling toward shore.

In the following it is convenient to introduce the functions

$$(8) \qquad \chi^+(z) = \tfrac{1}{2}(\chi^S(z) - i\chi^R(z)),$$

$$\chi^-(z) = \tfrac{1}{2}(\chi^S(z) + i\chi^R(z)).$$

Waves progressing toward shore are then given by

$$(9) \qquad \chi^P(z,t) = \chi^+(z)e^{it} + \chi^-(z)e^{-it} \ .$$

3. Our main problem is to find an asymptotic representation of the functions $\chi^R(z)$ and $\chi^S(z)$ for small values of the slope angle ω. There are three different asymptotic representations depending on which quantities are kept fixed while one lets ω approach zero.

The first, and most natural, problem of asymptotic representation arises when one asks for the asymptotic value of $\chi^R(z)$ and $\chi^S(z)$ at a fixed point z. Evidently, only points $z = x$ on the water surface remain in the water region for arbitrarily small angles ω ; therefore, we consider only points $z = x$. In accordance with the shallow water theory, see Stoker [3, Section 3], the answer* to this question is probably that $\chi^R(x)$ and $\chi^S(x)$ behave asymptotically like

$$(10)\quad \chi_1^R = 2\pi J_o(2\sqrt{x/\omega}), \quad \chi_1^S = 2\pi Y_o(2\sqrt{x/\omega}), \quad \omega \approx 0,$$

involving the regular and singular Bessel functions J_o and Y_o. These Bessel functions die out at infinity. Therefore the approximate solutions (10) cannot represent the exact solutions far away from shore since the wave motion does not die out but behaves sinusoidally at large distances from shore. For a slope angle $\omega = 6^o$, for example, the expressions

*The approximate solution for beaches with a small slope based on a power series development of the exact solutions, given by Miche [1, p.139], is essentially the same as that given by the shallow water theory.

(10) give a good approximation only in a region covering one to two wave lengths near shore, while the asymptotic representation to be discussed next gives nearly perfect agreement with the exact solution all along the water surface except on a section near shore covered by about one third of the length of the first wave, see Figures 2 and 3.

4. In the second asymptotic representation, which we shall derive in great detail, the product ωz is kept fixed while ω approaches zero. In other words, we ask for the asymptotic values of the functions $\chi(z)$ at places z which move away from shore so that their distance from shore is inversely proportional to the angle ω. Again we consider only points $z = x$ on the water surface. The asymptotic representations χ_2^+ and χ_2^- of χ^+ and χ^- are given by the expressions

$$(11) \qquad \chi_2^{\pm}(x) = -(1 \mp \lambda)\, A(\lambda) e^{\pm i\omega^{-1}k(\lambda) \pm i\frac{\pi}{4}},$$

in which λ is a function of ωx given by the inverse of the relation

$$(12) \qquad \omega x = \lambda \operatorname{Artanh} \lambda = \frac{1}{2}\lambda \ln \frac{1+\lambda}{1-\lambda}, \quad 0 < \lambda < 1.$$

$A(\lambda)$ is given by

$$(13) \qquad A(\lambda) = \sqrt{\frac{2\pi\omega}{j(\lambda)}},$$

and $k(\lambda)$, $j(\lambda)$ are given by

$$k(\lambda) = \operatorname{Artanh}\lambda + \int_0^{\lambda} \operatorname{Artanh}\nu \, \frac{d\nu}{\nu} , \tag{14}$$

$$j(\lambda) = (1-\lambda^2)\,\lambda k'(\lambda) = \lambda + (1-\lambda^2)\operatorname{Artanh}\lambda . \tag{15}$$

The behavior of the quantities k, λ, j, and $A/\sqrt{2\pi\omega}$ as functions of ωx is shown in Figures 4, 5, 6, and 7. The asymptotic representations of the complex potential functions $\chi^R(x)$ and $\chi^S(x)$ for <u>standing waves</u> are then derived from (8). The asymptotic representation of the complex potential function $\chi^P(x,t)$ for progressing waves traveling toward shore is found from (9),

$$\chi_2^P(x,t) = -(1-\lambda)A(\lambda)e^{i[\omega^{-1}k(\lambda)+t+\frac{\pi}{4}]} - (1+\lambda)A(\lambda)e^{-i[\omega^{-1}k(\lambda)+t+\frac{\pi}{4}]} . \tag{16}$$

Each of the two contributions here evidently represents a wave traveling toward shore without being a solution itself.

The <u>behavior of the asymptotic solutions</u> (11) <u>at infinity</u>, $\omega x \approx \infty$, is determined from the formulas

$$\lambda \approx 1 - 2e^{-2\omega x} , \tag{17}$$

$$k(\lambda) \approx \frac{\pi^2}{8} + \omega x, \quad j(\lambda) \approx 1 , \tag{18}$$

(19) $$A(\lambda) \sim \sqrt{2\pi\omega} \ ,$$

which can be derived from (12), (14), (13), and (15). Hence we have for $\omega x \approx \infty$

(20) $$\chi_2^+(x) \approx 0, \quad \chi_2^-(x) \approx 2C\, e^{-ix} \ ,$$

with

(21) $$C = -\sqrt{2\pi\omega}\, e^{-i(\omega - 1 \frac{\pi^2}{8} + \frac{\pi}{4})} \ ,$$

whence

(22) $$\chi_2^R(x) \approx \chi_\infty^R = -iC\, e^{-ix}, \ \chi_2^S(x) \approx \chi_\infty^S = C\, e^{-ix} \ ,$$

as $\omega x \approx \infty$,

while for the progressing wave given by (16),

(23) $$\chi_2^P(x,t) \approx \chi_\infty^P = 2C\, e^{-i(x+t)}, \text{ as } \omega x \approx \infty.$$

The behavior of the asymptotic solutions (11) for small values of ωx is determined from the formulas

(24) $$\lambda \approx \sqrt{\omega x} \ ,$$

(25) $$k(\lambda) \approx 2\sqrt{\omega x} \ , \quad j(\lambda) \approx 2\sqrt{\omega x} \ ,$$

$$(26) \qquad A(\lambda) \approx \sqrt{\pi}\ \sqrt[4]{\omega/x}\ ,$$

which are also easily derived from (12), (14), (15), and (13). Hence we have for $\omega x \approx 0$,

$$(27) \qquad \chi_2^{\pm}(x) \approx -\sqrt{\pi}\ \sqrt[4]{\omega/x}\ e^{\mp 2i\sqrt{x/\omega}\ \pm\ i\pi/4}\ ,$$

$$(28) \qquad \chi_2^{R}(x) \approx 2\sqrt{\pi}\ \sqrt[4]{\omega/x}\ \sin(2\sqrt{x/\omega}\ +\ \pi/4)$$

$$\chi_2^{S}(x) \approx -2\sqrt{\pi}\ \sqrt[4]{\omega/x}\ \cos(2\sqrt{x/\omega}\ +\ \pi/4),$$

$$(29) \qquad \chi_2^{P}(x,t) \approx -2\sqrt{\pi}\ \sqrt[4]{\omega/x}\ \cos(2\sqrt{x/\omega}\ +\ t\ +\ \pi/4).$$

We note that both asymptotic solutions χ_2^R and χ_2^S have at $\omega x = 0$ the same singularity, that of $\sqrt[4]{\omega/x}$, which thus overshadows the logarithmic singularity of the exact solution χ^S and the regularity of the exact solution χ^R. The reason for this discrepancy is that the accuracy with which the asymptotic solutions (11) represent the exact solutions decreases as x approaches zero for a fixed value of ω while it increases as ω approaches zero for a fixed value of ωx.

It is quite illuminating in this respect that the functions (28), representing the "second" asymptotic solutions (11) for small values of ωx, at the same time represent the "first" asymptotic solutions (10) for large values of x, as follows from the well known

asymptotic expansions of the Bessel functions for large values of the argument. For a slope angle $\omega = 6^{\circ}$ the expressions (28) represent the solutions quite well in a range covering two-thirds of the first wave and most of the second wave, see Figures 2 and 3. In this range, therefore, the "first" and the "second" asymptotic representations agree.

The surface elevations of the standing waves are by (1), (2), (8), and (11) given asymptotically as

$$(30)\quad \eta_2^R(x,t) = -2A(\lambda)\sin(\omega^{-1}k(\lambda) + \tfrac{\pi}{4})\sin t$$

$$\eta_2^S(x,t) = 2A(\lambda)\cos(\omega^{-1}k(\lambda) + \tfrac{\pi}{4})\sin t.$$

As "local" wave length for these surface waves we may introduce the quantity

$$(31)\qquad \Lambda = 2\pi[\frac{d}{dx}(\omega^{-1}k(\lambda) + \tfrac{\pi}{4})]^{-1} = 2\pi\, \frac{d\omega x}{d\lambda}/\frac{dk}{d\lambda}$$

because this expression would be the wave length if $\omega^{-1}k(\lambda)$ were proportional to x and the wave accordingly purely sinusoidal. From (12) and (14) we find $d\omega x/d\lambda = \lambda k'(\lambda)$; hence the local wave length is just

$$(32)\qquad \Lambda = 2\pi\lambda .$$

The coefficient $2A(\lambda)$ of the sinusoidal functions in the expressions (30) for the surface elevation may be called the "amplitude" as function of the wave

length. It is remarkable that this amplitude, which according to (19) approaches the value $\sqrt{2\pi\omega}$ as $\omega x \to \infty$, decreases as ωx decreases until it reaches the minimum

$$(33) \qquad 2A_{min} = .915\sqrt{\frac{2}{\omega\pi}} \quad \text{at} \quad \omega x = 1,$$

and then increases indefinitely as ωx approaches zero, see Figure 7. This fact, already observed by Rankine, see Miche [1, p.287] is easily derived from (12), (13), and (15). It is in agreement with the behavior of the exact solution, see [3].

For the surface elevation of the progressive wave we obtain from (16) and (2)

$$(34) \qquad \eta_2^P(x,t) = 2A(\lambda)\sin(\omega^{-1}k(\lambda) + t + \tfrac{\pi}{4}).$$

As for standing waves we can interpret the quantity $2\pi\lambda$ as the "local" wave length and $2A(\lambda)$ as the "amplitude" as function of the wave length. The expression (13) for this amplitude agrees with that derived by Miche [1,p286] for progressing waves on the basis of energy flux considerations.

5. We now proceed to discuss the third asymptotic problem. It arises if one considers as fixed the distances in x-direction from that cross section $x = x_o$ at which the depth of the water has a prescribed value h_o. Thus the shore line recedes to $-\infty$ as $\omega \to 0$. If the angle ω is small we may set $\omega x = h$. Hence the cross section $x = x_o$ is given by

(35) $$x_o = \omega^{-1} h_o \ .$$

Accordingly, we keep the values $z - x_o = z - \omega^{-1} h_o$ fixed. The asymptotic expressions of the third kind for the functions $\chi(z)$ at the top $z = x_o$ of the cross section $x = x_o$ are obviously just the same as the asymptotic expressions of the second kind. The asymptotic expressions of the third kind at a point $z = x$ with fixed $x - x_o$ can also be derived from those of the second kind; for, a fixed value of $x - x_o$ corresponds to values of $\omega x - \omega x_o = \omega x - h_o$ which approach zero as ω approaches zero. Hence the first terms of the expansion of the expressions of the second kind with respect to powers of $\omega x - h_o$ give the desired asymptotic expressions of the third kind. By virtue of

(36) $$\frac{d}{d\omega x} k(\lambda) = \lambda^{-1} \ ,$$

see (31) and (32), we have

(37) $$k(\lambda) = k(\lambda_o) + \lambda_o^{-1}(\omega x - h_o) + \dots \ ,$$

the value λ_o being determined from

(38) $$h_o = \lambda_o \operatorname{Artanh} \lambda_o$$

according to (12) and (35). Consequently, we find from (11) the expression

(39) $$\chi_3^{\pm}(x) = -(1 \mp \lambda_o) A(\lambda_o) e^{\pm i \omega^{-1} k(\lambda_o) \pm i \frac{\pi}{4}} e^{\pm i \lambda_o^{-1}(x - x_o)}$$

as asymptotic representation of the third kind of the functions $\chi^{\pm}(z)$ at the top surface $z = x$. It is plausible that the asymptotic representation of the third kind all through the water depth near $z = x_o$ is given by

$$(40)\quad \chi_3^{\pm}(z) = -(1 \mp \lambda_o)A(\lambda_o)e^{\mp i\omega^{-1}k(\lambda_o)\pm i\frac{\pi}{4}}\, e^{\mp i\lambda_o^{-1}(z-x_o)}\,,$$

$$-h_o \le y \le 0\,.$$

For the standing waves we find

$$(41)\quad \chi_3^{S,R}(z) = (1-\lambda_o)C^{S,R}\, e^{i\lambda_o^{-1}(z-x_o)} + (1+\lambda_o)\overline{C^{S,R}}\, e^{-i\lambda_o^{-1}(z-x_o)},$$

valid for $-h_o \le y \le 0$, with

$$(42)\qquad C^S = C_o\,,\quad C^R = iC_o\,,$$

$$(43)\qquad C_o = A(\lambda_o)e^{i\omega^{-1}k(\lambda_o)+i\frac{\pi}{4}}\,.$$

The standing waves (41) are evidently superpositions of two standing waves with the wave length $2\pi\lambda_o$; as a matter of fact, they are just Airy's solutions for the flow in a channel of constant depth h_o. They satisfy the conditions

$$(44) \qquad \mathrm{Im}\left(\frac{d}{dz} + i\right)\chi_3 = 0 \quad \text{on} \quad y = 0$$

$$\mathrm{Im}\,\frac{d}{dz}\,\chi_3 = 0 \quad \text{on} \quad y = -h_o\ ,$$

as is easily verified. Relation (38) is just Airy's relation between wave length $2\pi\lambda_o$ and depth h_o.

The question has frequently arisen whether or not it is correct to employ Airy's relation, derived for various constant depths, to describe also the variation of the wave length on a sloping beach. Our result shows that it is indeed correct to do so approximately, but with good accuracy, for small slope angles.

The asymptotic expression of the third kind for progressing waves traveling toward shore is derived from (8), (9), and (41)

$$(45) \qquad \chi_3^P(z,t) = -(1 - \lambda_o)C_o e^{i\left\{\lambda_o^{-1}(z-x_o)+t\right\}}$$

$$+ (1 + \lambda_o)\bar{C}_o e^{-i\left\{\lambda_o^{-1}(z-x_o)+t\right\}}\ .$$

Both contributions are again waves traveling toward shore, though not solutions individually. The difference in the two contributions lies in this - that the amplitude of the first contribution $\chi_3^+(z)e^{it}$ increases like e^{-y} from the top surface to the bottom while that of the second contribution $\chi_3^-(z)e^{-it}$ decreases like e^{y}. Note that for infinite x the first contribution disappears and the second one occurs with the amplitude

$2\sqrt{2\pi\omega}$, see (20), (21), while at the shore both contributions occur with equal amplitude, see (27). The expression (45) of the progressing wave with $|C_o| = A(\lambda_o)$, except for the dependence of the phase on the depth, contained in the expression (43) for C_o, was found by Miche [1 p.282] on the basis of considering the energy flux through any cross section. Our investigation now shows that this formula is the correct asymptotic expression for small values of the beach angle.

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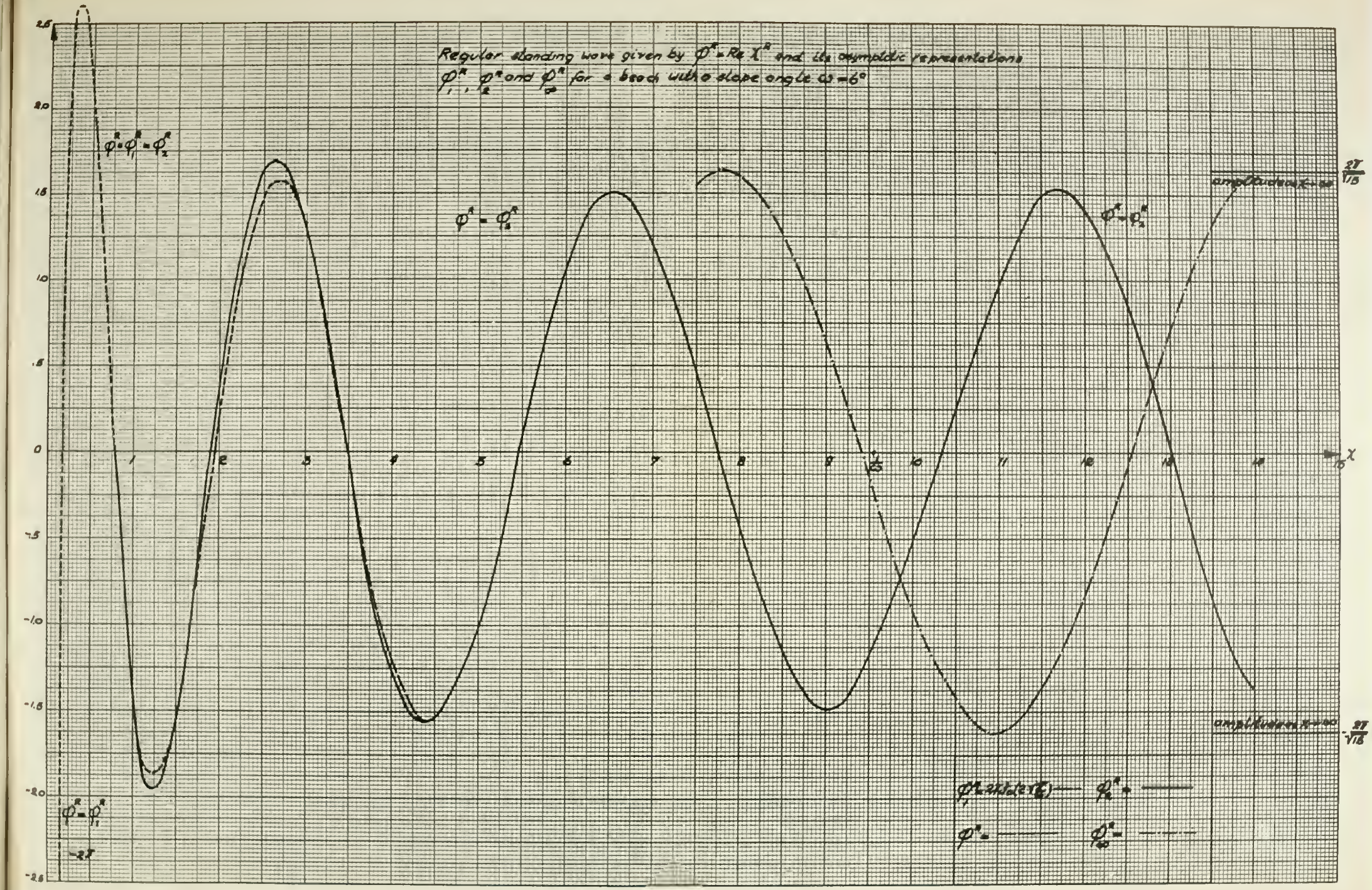
Regular standing wave given by $\varphi^R = \mathrm{Re}\, X^R$ and its asymptotic representations
φ_1^R, φ_2^R and φ_∞^R for a beach with a slope angle $\alpha = 6°$
$\varphi^R = \varphi_1^R = \varphi_2^R$
$\varphi^R = \varphi_\infty^R$
$\varphi^R = \varphi_2^R$
$\varphi^R = \varphi_1^R$
amplitude $X = \infty$
$\frac{2\Gamma}{\sqrt{18}}$
amplitude $X = \infty$
$-\frac{2\Gamma}{\sqrt{18}}$
-2Γ
φ_1^R + 2 (dots) φ_2^R + ——
φ^R = —— φ_∞^R = - - -
2.5
2.0
1.5
1.0
.5
0
-.5
-1.0
-1.5
-2.0
-2.5
1
2
3
4
5
6
7
8
9
10
11
12
13
14
$\frac{\pi}{6}$
χ

Figure 2

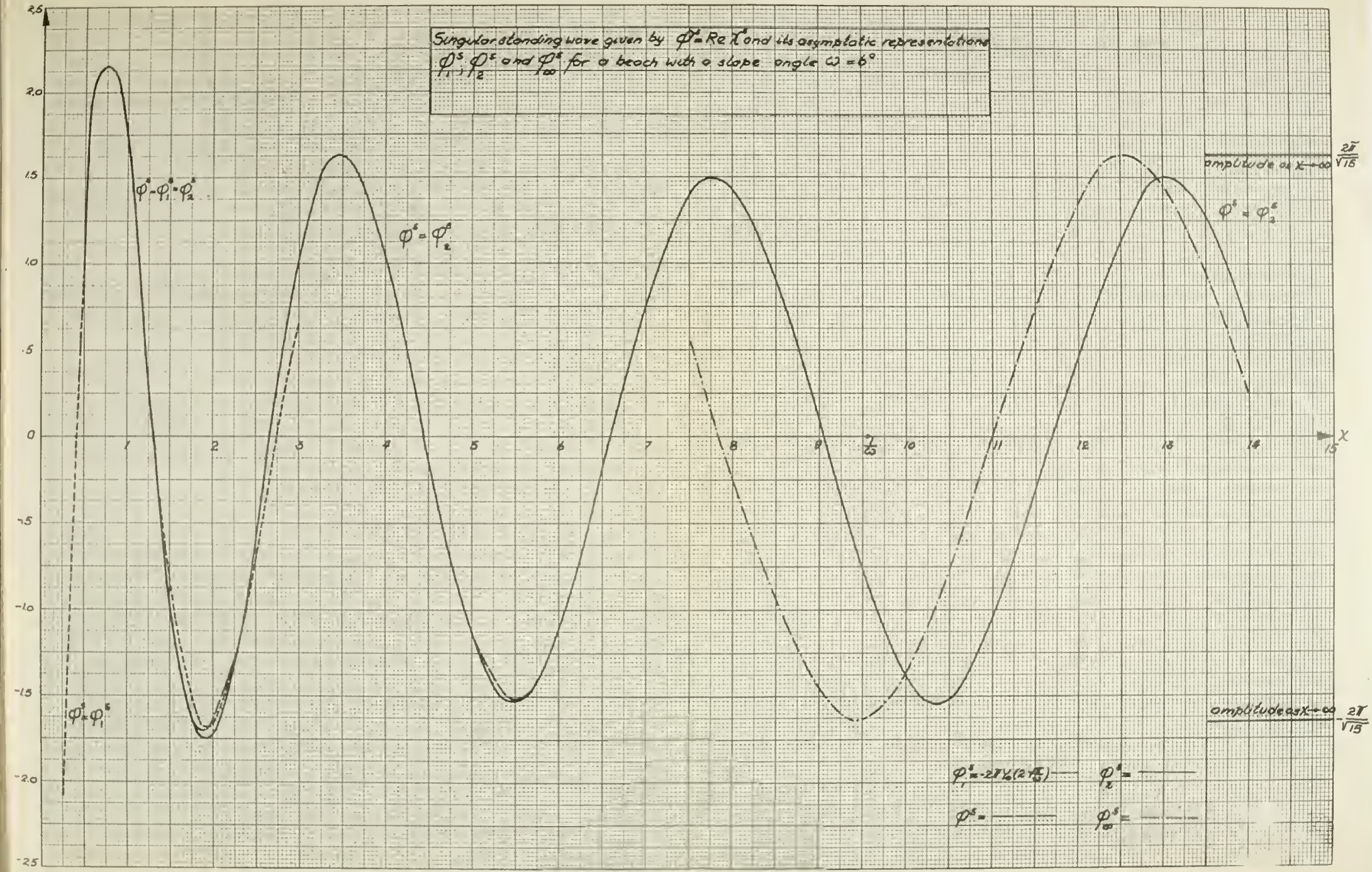
Singular standing wave given by $\varphi^s = Re\,\bar{\chi}$ and its asymptotic representations φ_1^s, φ_2^s and φ_∞^s for a beach with a slope angle $\omega = 6°$
$\varphi^s = \varphi_1^s + \varphi_2^s$
$\varphi^s = \varphi_2^s$
$\varphi^s = \varphi_1^s$
$\varphi^s = \varphi_2^s$
amplitude as $x \to -\infty$ $\frac{2\pi}{\sqrt{15}}$
amplitude as $x \to -\infty$ $\frac{2\pi}{\sqrt{15}}$
$\varphi_1^s = -2\pi Y_0(2\pi\sqrt{x})$ —— $\varphi_2^s =$ ——
$\varphi^s =$ —— $\varphi_\infty^s =$ ——
2.5
2.0
1.5
1.0
.5
0
-.5
-1.0
-1.5
-2.0
-2.5
1
2
3
4
5
6
7
8
9
$\frac{7}{25}$
10
11
12
13
14
15
x

Figure 3

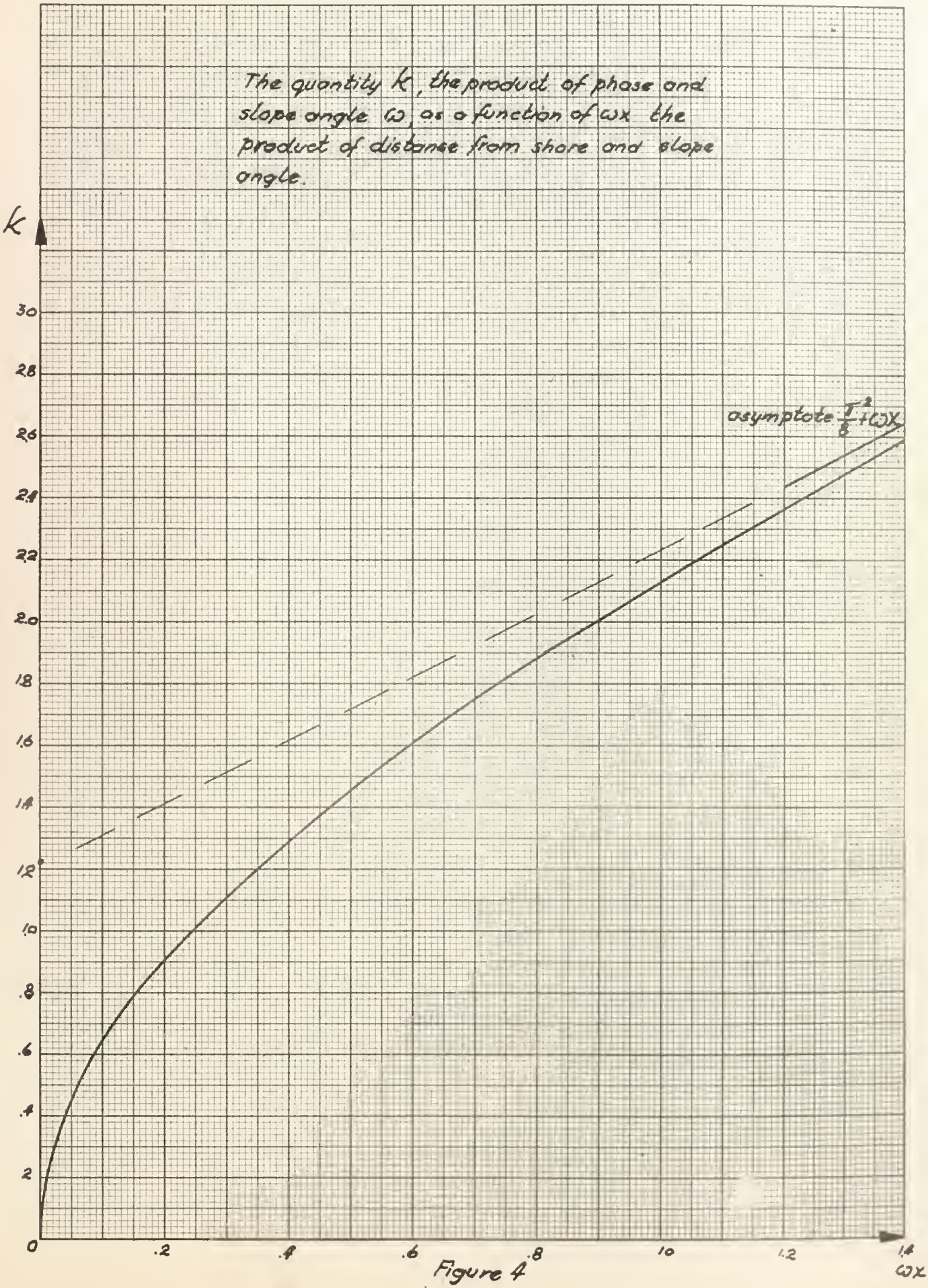
The quantity K, the product of phase and slope angle ω, as a function of ωx the product of distance from shore and slope angle.
K
30
28
26
24
22
20
18
16
14
12
10
.8
.6
.4
2
0
asymptote $\frac{\pi}{8}+\omega x$
.2
.4
.6
.8
10
1.2
1.4
ωx

Figure 4

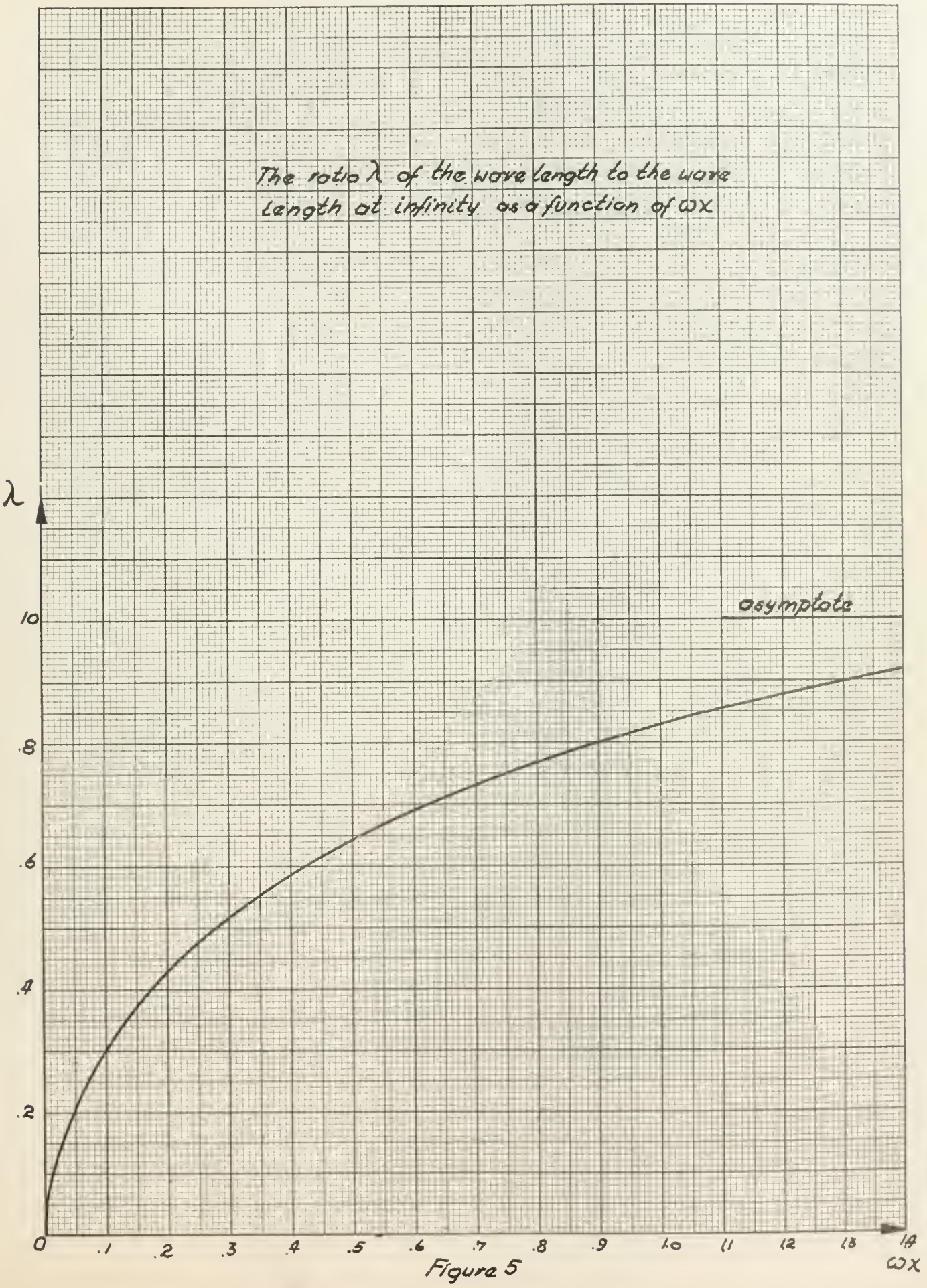
The ratio λ of the wave length to the wave length at infinity as a function of ωx
λ
asymptote
10
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
ωx

Figure 5

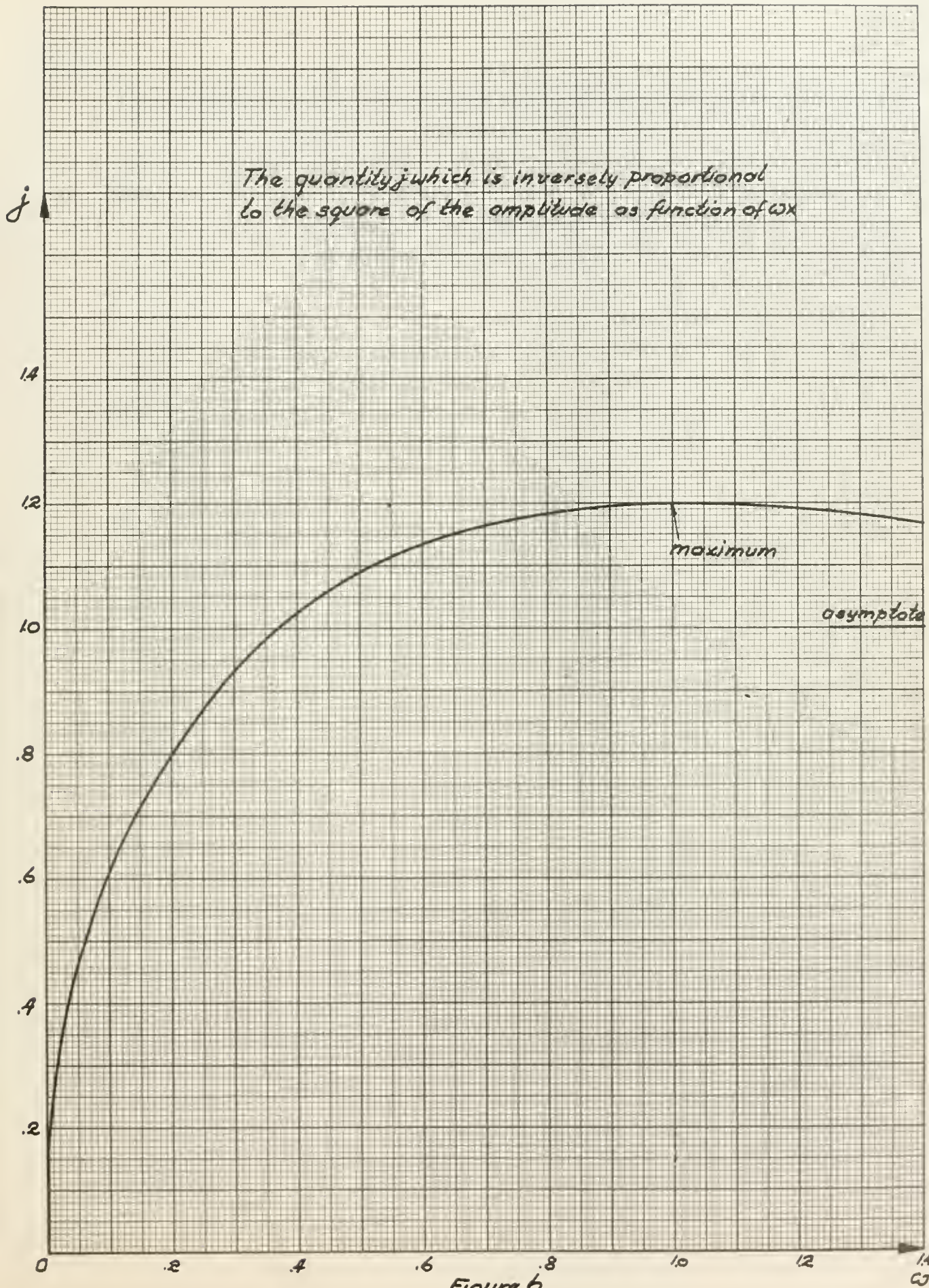
The quantity j which is inversely proportional
to the square of the amplitude as function of ωx
j
1.4
1.2
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
1.2
1.4
ωx
maximum
asymptote

Figure 6

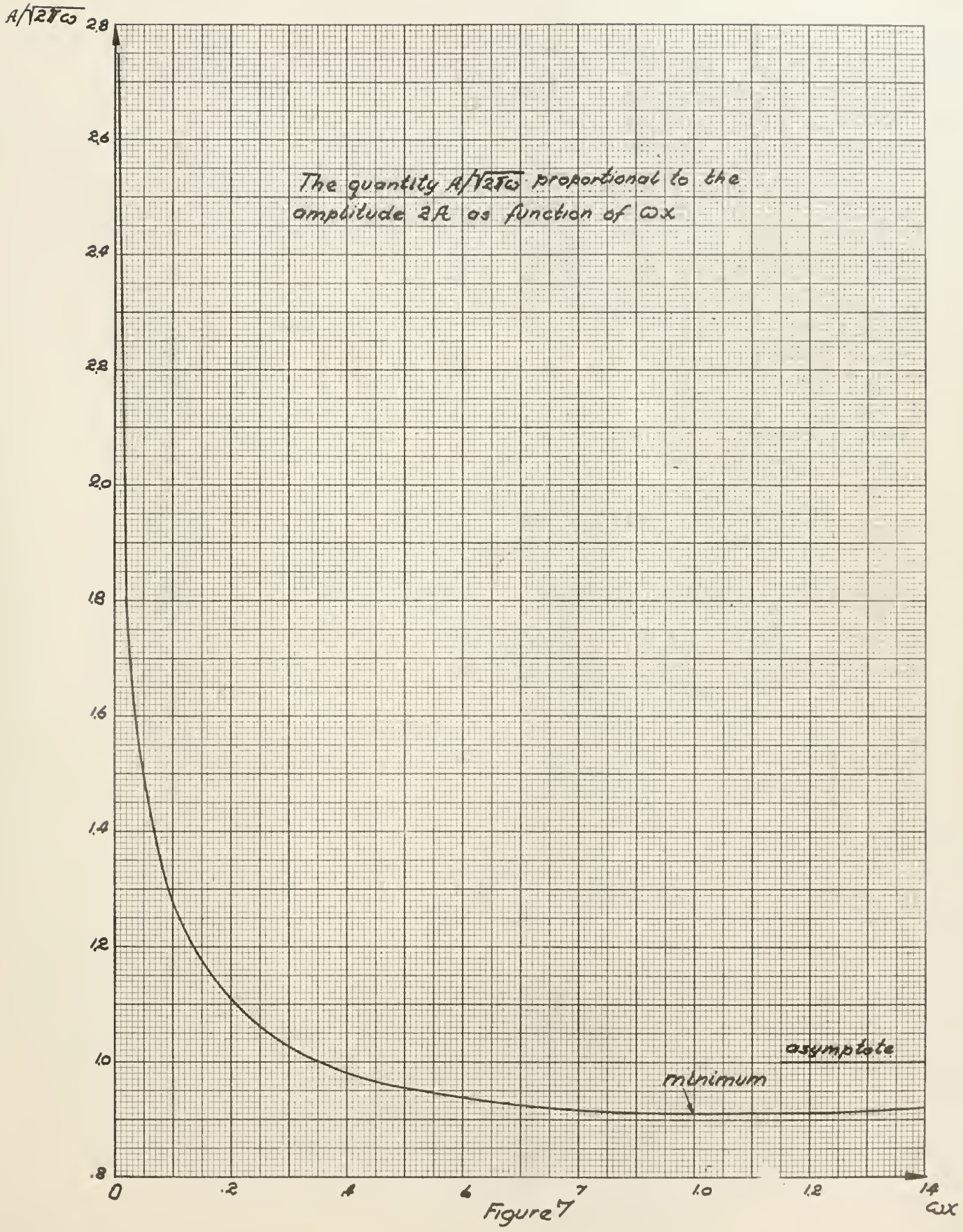
A/√2πω
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
The quantity A/√2πω proportional to the amplitude 2A as function of ωx
asymptote
minimum
0
.2
.4
.6
.7
1.0
1.2
1.4
ωx

Figure 7

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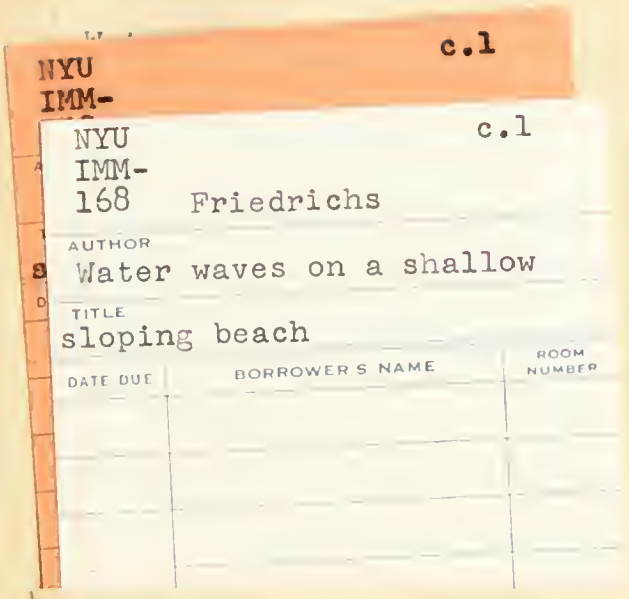

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